A Note on Primes Dividing Alternating Sums

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We are all familiar with the harmonic sum:

$$S_n = \sum_{i=1}^{n} \frac{1}{i}$$

which can easily be shown not to be an integer unless $n = 1$ (see [1], chapter 1, exercise 30).

On the other hand we can consider the alternating sum:

$$A_n = \sum_{i=1}^{n} (-1)^{i-1}\frac{1}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}$$

which is not an integer either. Nevertheless we will be able to write $A_n = \dfrac{a}{b}$ with a and b being coprime integers.

Let us start by considering an odd integer n such that $\dfrac{3n+1}{2} = p$ is a prime. Note that in such a case it is easily seen that n must satisfy $n \equiv 3 \pmod 4$. Now we construct the sum $A_n = \dfrac{a}{b}$ and we claim p divides a.

In fact we can write

$$A_n = S_n - 2\left(\sum_{i=1}^{\frac{n-1}{4}} \frac{1}{2i}\right) = S_n - S_{\frac{n-1}{2}} = \frac{1}{\frac{n-1}{2}+1} + \frac{1}{\frac{n-1}{2}+2} + \cdots + \frac{1}{n}$$

Of course, as p is a prime bigger than n, we see that the numbers $\dfrac{n-1}{2} + k$ are, all of them, units in $\mathbb{Z}/p\mathbb{Z}$ (observe that $k = 1, \ldots, \frac{n+1}{2}$). Moreover, there is an even number of summands in $S_n - S_{\frac{n-1}{2}}$. Now, if we work modulo p and we choose any $k \in \{1, \ldots, \frac{n+1}{2}\}$ we find that $\dfrac{n-1}{2} + k + n - k + 1 = p$ so $\dfrac{1}{\frac{n-1}{2}+k} \equiv \dfrac{-1}{n-k+1} \pmod p$ and we get that $A_n \equiv 0 \pmod p$ as claimed.

Now, if we choose an even number n such as $\dfrac{3n+2}{2} = p$ is a prime, we can reason in the same way and conclude that $A_n \equiv 0 \pmod p$. Note that, in this case, it must be $n \equiv 0 \pmod 4$)

Finally, we may reformulate the preceding results in the following way:

Theorem 1. *Let p be and odd prime. Then there exist an integer n such that $A_n \equiv 0 \ (mod\ p)$.*

Proof. Given an odd prime p it is easy to see that the number $2p - 1$ must be $2p - 1 \equiv 0, 1 \pmod 3$, i.e., it must be $2p - 1 = 3n$ or $2p - 1 = 3n + 1$ so it is enough to consider the corresponding A_n. □

References

[1] **Apostol, T. M.** *Introduction to analytic number theory*, Springer-Verlag, New York-Heidelberg, 1976